likely to be found among the electrolytes; this is indicated by the three studied by us, and also by those studied by McBain and Laing-McBain.[5]

Detergents have long been used for the dispersal of various types of substances of biological interest, particularly the proteins. It is important to emphasize that the detergent may not only affect the proteins studied but that some of its properties such as the molecular size may fall within that range usually considered to be characteristic of proteins alone.

*Summary.*—Ultracentrifugal observations using a direct reading refractive index method have been made on aqueous solutions of digitonin. Practically all of the digitonin exists in the form of micelles of homogeneous size, with an average sedimentation constant of $5.88 \times 10^{-13}$ cm. per dyne per sec. The micellar weight is likely to lie within the range of 75,000 to 400,000, as contrasted with a molecular weight of 1228.

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[4] L. G. Longsworth, *Jour. Am. Chem. Soc.*, **61**, 529 (1939).

[5] J. W. McBain, and M. E. Laing-McBain, *Proc. Roy. Soc. London*, **A, 139**, 26 (1933).

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# *A SET OF POSTULATES FOR BOLYAI-LOBATCHEVSKY GEOMETRY*

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1. *Introduction.*—In some recent papers[1] Menger proved that all concepts of the Bolyai-Lobatchevsky geometry can be defined in terms of the operations "joining" and "intersecting," basic to his algebra of projective and affine geometry, as well as to G. Birkhoff's lattice theory. It follows that a complete foundation of non-Euclidean geometry can be given in terms of these two concepts, e.g., by substituting into the ordinary postulates Menger's definitions of the concepts "between," "parallel," "congruent," etc., in terms of joining and intersecting. Since postulates obtained in this way would be very cumbersome, there arose the problem of establishing some simple direct postulates concerning the two operations

from which the non-Euclidean geometry could be developed. In what follows we give a list of eight postulates from which the whole theories of order and parallelism in non-Euclidean geometry can be derived.

We restrict ourselves to the case of the plane and use the concepts of "point," "line" and "lies on," which as is well known can be derived from the operations of joining and intersecting. Our present definitions of betweenness and parallelism differ from those originally given by Menger in that they avoid the use of "maximal triangles," and are stated directly in terms of the primitive notions.

2. *The Postulates.*—The following postulates will be assumed.

I. If $A$ and $B$ are any two distinct points, there exists exactly one line $m$ such that both $A$ and $B$ lie on $m$.

II. Each line contains at least five distinct points.

III. There exist at least three non-collinear points.

IV. If $a$, $b$ are any two distinct interesecting lines, and $P$ is a point not on $a$ or $b$, then there exists at least one line through $P$ which intersects $a$, but not $b$.

V. If $a$, $b$ are any two non-intersecting lines, and $P$ is any point not on $a$ or $b$, then there exists a line through $P$ which intersects neither $a$ nor $b$.

VI. If $A$, $B$, $C$ are distinct collinear points, and if there exist lines $a$ and $a'$ through $A$, $b$ and $b'$ through $B$, $c$ and $c'$ through $C$, such that

1) $b$ and $c$ intersect, but neither meets $a$,

2) $a'$ and $b'$ intersect, but neither meets $c'$,

then each line through $B$ meets at least one line of every pair of intersecting lines which pass through $A$ and $C$, respectively.

VII. If $a$, $b$, $c$ are three mutually non-intersecting lines, and if there exists a line meeting both $a$ and $b$, but not $c$, and also a line meeting $b$ and $c$, but not $a$, then through any point of $a$, there exists a line meeting $b$ but not $c$.

VIII. If $a$ is a given line, then through any point not on $a$ there exist at least two distinct lines which do not intersect $a$.

Postulate VIII is the only one which is not satisfied equally in the Euclidean plane, VI and VII being vacuously satisfied there. All but IV and VIII are true even in the projective plane.

3. *The Theory of Order.*—We say of three distinct points $A$, $B$, $C$ that $B$ lies *between* the two other points $A$ and $C$, if every line through $B$ intersects at least one line of each pair of intersecting lines which pass through $A$ and $C$, respectively. We show that any three points satisfying this definition must be collinear, and that the triadic relation so defined satisfies the conditions for a betweenness relation given by Huntington and Kline.[2] Moreover, we derive the statement known as the axiom of Pasch: If a line meets one side of a triangle in an interior point, then it meets exactly one of the other two sides in an interior point, or passes through the op-

posite vertex. Finally, the plane is shown to be convex and externally convex, i.e., to each pair of points $A$, $C$, there exist two points $B$, $D$ such that $B$ lies between $A$ and $C$, and $C$ lies between $A$ and $D$. These results imply that each line separates the plane into two parts.

4. *The Theory of Parallelism.*—Two non-intersecting lines $a$ and $b$ are said to be *parallel* if there exists a point $P$ such that through $P$ there is at most one line which meets neither $a$ nor $b$. This definition is proved to be independent of $P$ in the sense that each point lying between $a$ and $b$ has the same property as $P$. Here we say that the point $Q$ lies between the non-intersecting lines $m$ and $n$ if there exists a line through $Q$ which intersects $m$ and $n$ in points $M$ and $N$, respectively, such that $Q$ lies between $M$ and $N$. The relation of parallelism is clearly symmetric, and has the property that if $a$ is a parallel to $b$ through a point $Q$, and if $R$ is any other point of $a$, then $a$ is also a parallel to $b$ through $R$.

Let $c$ be a line which intersects two parallels $a$, $b$ in the points $A$, $B$, respectively, and $P$ be a point of $a$, distinct from $A$. We say that $a$ and $b$ are parallel on the side of $c$ on which $P$ lies, if there exists a line through $P$ intersecting $c$ in a point between $A$ and $B$, but not intersecting $b$. Here again the definition is shown to be independent of the point $P$, for we prove that any point of $a$ or $b$ which is on the side of $c$ on which $P$ lies may replace $P$ in the definition. Of course, for a line $c$ the lines $a$ and $b$ may be parallel on the side of $c$ on which $P$ lies, whereas for another line $d$, they might be parallel on the side of $d$ opposite to that on which $P$ lies.

If $a$ and $b$ are parallel, and $b$ and $c$ are parallel, and $t$ is a line intersecting each of the lines $a$, $b$, $c$, then $a$ and $c$ are said to be parallel to the line $b$ on the same side of the transversal $t$ if there exists a point $P$ on $b$ such that both $a$ and $b$, and $b$ and $c$ are parallel on that side of $t$ on which $P$ lies.

Using these concepts, we prove the classical assumption that the relation of parallelism is transitive in a certain sense: If $a$ and $c$ are two lines which are parallel to $b$ on the same side of a transversal, then $a$ is parallel to $c$. Further, if $c$, $d$ are two parallels to a line $m$ through a point $P$, then $c$ and $d$ are parallel to $m$ on opposite sides of any line through $P$ meeting $m$, and hence there exist at most two parallels to any given line through a given point.

For the full development of the above results, see the author's papers in the *Reports of a Mathematical Colloquium*, Issue 1, pp. 45–48, Issues 2 and 3 in press.

[1] *Proc. Nat. Acad. Sci.*, **24**, 486–490 (1938); *Compt. Rend., Paris*, **207**, 458–460 (1938); *Bull. Amer. Math. Soc.*, December, 821–824 (1938).

[2] *Trans. Amer. Math. Soc.* **18**, 301–325 (1917).